16.2: Gradient fields
16.4 Green's Theorem

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Last class

Definition

A vector field is a function

$$\vec{\mathbf{F}} = M(x,y,z)\vec{\mathbf{i}} + N(x,y,z)\vec{\mathbf{j}} + P(x,y,z)\vec{\mathbf{k}}$$

that outputs a vector for every point (x,y,z) in space.

Definition

Let $\vec{\mathbf{F}}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$. Then the line integral of $\vec{\mathbf{F}}$ along C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}) dt.$$

Gradient fields

Definition

The gradient field of a differentiable function $f(x, y, z)$ is the vector field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\vec{\mathbf{i}} + \frac{\partial f}{\partial y}\vec{\mathbf{j}} + \frac{\partial f}{\partial z}\vec{\mathbf{k}}$$

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Example

Find the gradient field of $g(x, y, z) = e^z - \ln(x^2 + y^2)$.

Taking partial derivatives, we find that the gradient field of g is

$$\nabla g = \langle -\frac{2x}{x^2 + y^2}, -\frac{-2y}{x^2 + y^2}, e^z \rangle.$$

Green's Theorem in the plane

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However, if we take $0 \leq t \leq \pi$ above, it is only simple, and if we take $0 \leq t \leq 4\pi$, it is closed but not simple.

Integral notation

If we take a line integral over a closed curve C, we use the notation

$$\oint_C$$

to represent the fact that the line integral comes back to where it started.

Green's Theorem

Theorem

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{\mathbf{F}} = M(x, y)\vec{\mathbf{i}} + N(x, y)\vec{\mathbf{j}}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the following holds.

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

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This says we can calculate the line integral of a vector field over a closed curve as a double integral over the region that the curve encloses.

Example

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Let $\vec{\mathbf{F}} = (x^2y)\vec{\mathbf{i}} + (xy^3)\vec{\mathbf{j}}$*. Let* C *be the closed triangular curve beginning at (0,0), going to (1,0), then to (0,1), then back to (0,0). Calculate* $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ *using Green's Theorem.*

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We parametrize C as three lines:

$$\vec{\mathbf{r}}_1(t) = t\vec{\mathbf{i}},\ \ \vec{\mathbf{r}}_2(t) = (1-t)\vec{\mathbf{i}} + (t)\vec{\mathbf{j}},\ \ \vec{\mathbf{r}}_3(t) = (1-t)\vec{\mathbf{j}} \text{ each over } 0 \leq t \leq 1$$

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Notice that $\vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) = \vec{\mathbf{0}}$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) = \vec{\mathbf{0}}$. Thus

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Notice that $\vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) = \vec{\mathbf{0}}$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) = \vec{\mathbf{0}}$. Thus

$$\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}} ds = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t))\cdot\frac{d\vec{\mathbf{r}}_2}{dt} dt = \int_0^1 \langle (1-t)^2(t), (1-t)t^3\rangle\cdot\langle -1, 1\rangle dt.$$

$$= \int_0^1 (-t+2t^2-t^3+t^3-t^4) dt = -\frac{t^2}{2}+\frac{2}{3}t^3-\frac{t^5}{5}\Big]_0^1 = -\frac{1}{2}+\frac{2}{3}-\frac{1}{5} = -\frac{1}{30}$$

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Let $\vec{\mathbf{F}} = (x^2y)\vec{\mathbf{i}} + (xy^3)\vec{\mathbf{j}}$. Let C be the closed triangular curve beginning at (0,0), going to (1,0), then to (0,1), then back to (0,0). Calculate $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ using Green's Theorem.

We could also calculate the line integral using Green's Theorem. We have $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y^3 - x^2$, and the triangle formed by $(0,0)$, $(1,0)$, and $(0,1)$ can be written with horizontal cross-sections as $0 \le x \le 1 - y,\ 0 \le y \le 1$. Thus

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} (y^3 - x^2)\, dx\, dy = \int_{y=0}^{y=1} \left[y^3 x - \frac{x^3}{3} \right]_{x=0}^{x=1-y} dy$$

$$= \int_0^1 (y^3 - y^4 - \frac{1}{3} + \frac{y^3}{3} + y - y^2) dy = \frac{1}{4} - \frac{1}{5} - \frac{1}{3} + \frac{1}{12} + \frac{1}{2} - \frac{1}{3} = -\frac{1}{30}$$